Quantum Entropy Bound by Information in Black Hole Spacetime

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We show that the increase of the generalized entropy by a quantum process outside the horizon of a black hole is more than the Holevo bound of the classical information lost into the black hole and which could be obtained by further observations outside the horizon. In the optimal case, the prepared information can be completely retrieved.

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I. INTRODUCTION

Bekenstein [1], on the basis of information theoretical arguments in a gedanken experiment, proposed the generalized second law in the black hole spacetime prior to the discovery of the Hawking radiation [2] and thus opened up black hole thermodynamics [3]. It has been shown that there is an almost complete parallelism between black hole physics and thermodynamics from the zero-th to the third law. However, there remains a long standing problem: the apparent loss of information about the initial state by evaporation of the black hole [4]. From our point of view, it is crucial to clarify the meaning of "information" to resolve this paradox. Recently, the information theoretical aspects in black hole physics have been reemphasized [5] in the light of the entropy bound conjecture.

In the black hole thermodynamics the total entropy is the sum of the black hole entropy $S_{BH} = A/4$ (where A is the area of the black hole horizon, and $S_{BH} = 4\pi M^2$ for a spherical black hole of mass M) and of the ordinary matter entropy S_M, i.e. $S_T = S_{BH} + S_M$. The generalized second law is motivated by the paradox of Wheeler's demon: although the entropy S_M of the matter outside the black hole decreases by disposing it to the black hole, the total entropy ΔS_T increases. There is plenty of evidence to support it. For example, a gedanken experiment suggested by Unruh and Wald [6] takes into account the Unruh effect [7,8], while Frolov and Page [9] gave a general argument based on the EPR-like entanglement of the particle states inside and outside the event horizon. In a previous work [10] the present authors showed that, in a *quantum* version of the Geroch-Bekenstein gedanken experiment, for the outside region of a black hole the total entropy increases, while the matter entropy decreases when a detector is dropped into the black hole. The decrease of the matter entropy is more than compensated by the increase of the black hole entropy via the increase of the black hole mass which is ultimately attributed to the work done by the measurement. In the present work we will show further that the increase of the generalized entropy is greater than or equal to the Holevo bound [11,12], which in turn is the upper bound to the classical information which can be obtained by quantum measurements. Entanglement plays an essential role in our argument and is a key concept of quantum information theory [13].

II. QUANTUM ENTROPY BOUND

The quantum state of the matter in the black hole spacetime is described by the Hartle-Hawking state,

$$|\psi>_{HH} \equiv \sum_n \sqrt{c_n} |n>_B \, |n>_A, \qquad (1)$$

where $c_n \equiv \exp[-\omega n/T_{BH}]/Z$ is the Boltzmann factor, $Z \equiv \sum_n \exp[-\omega n/T_{BH}]$ and $T_{BH} \equiv (8\pi M)^{-1}$ is the Hawking temperature. The state (1) is an entangled state [9] of the particles inside ($|n>_B$) and outside ($|n>_A$) of the black hole just like the EPR pair (for a review see, e.g., [13]). The state inside the black hole is not accessible from the outside so that we trace over the B-state to obtain a mixed state for the observer outside, i.e. $\rho_A \equiv \mathrm{Tr}_B(|\psi>_{HH}<\psi|) = \sum_n c_n |n>_A<n|$, which is nothing but the canonical thermal density operator [14]. Now imagine a detector of negligible mass in the pure state $|\Phi_0>$, initially located far away from the black hole horizon, which is slowly lowered by a string up to a point near the horizon, and then a quantum experiment outside of the black hole is performed. The reduced A state will change in general as

$$\rho_A \to \rho'_A \equiv \sum_\alpha A_\alpha \rho_A A^\dagger_\alpha = \sum_\alpha p_\alpha \rho'_\alpha, \qquad (2)$$

with $\sum_\alpha A^\dagger_\alpha A_\alpha = 1$. The transition is represented by a trace preserving positive operator valued measurement (POVM), where $p_\alpha \equiv \mathrm{Tr}\,(A_\alpha \rho_A A^\dagger_\alpha)$ is the probability to get the measurement result α, and $\rho'_\alpha \equiv (A_\alpha \rho_A A^\dagger_\alpha)/p_\alpha$ is the new normalized density operator. The POVM process is more physically understood if we explicitly introduce detector states $|\Phi_\alpha>$ tensored to the entangled state (1). In more details, when the agent outside the black hole switches on his experimental apparatus, the system will undergo a unitary transformation U for the compound state of A and the detector as

$$|\Psi >\rightarrow |\Psi >', \tag{3}$$

where

$$\begin{aligned} |\Psi > &\equiv \sum_n \sqrt{c_n}|n>_B \, |n>_A \, |\Phi_0(x_0)> \\ |\Psi >' &\equiv \sum_n \sqrt{c_n}|n>_B \, U(|n>_A \, |\Phi_0(x_0)>) \\ &= \sum_{\alpha,n} \sqrt{c_n}|n>_B \sum_m U^\alpha_{nm}|m>_A \, |\Phi_\alpha(x_0)>, \end{aligned} \tag{4}$$

and where x_0 is the spacetime point of the detector, which is initially located outside the horizon. We assume that by the measurement the state decoheres (on a proper timescale which ensures that the process is quasi-static, and which is smaller than the dynamical timescale of the process itself) to a diagonal form with respect to the detector states $|\Phi_\alpha(x_0)>$. The resultant mixed state ρ' is then

$$\begin{aligned} \rho' = \sum_\alpha & \left(\sum_n \sqrt{c_n}|n>_B \sum_m U^\alpha_{nm}|m>_A\right) \\ & \cdot \left(\sum_{n'} \sqrt{c_{n'}}\,{}_B<n'| \sum_{m'} U^{*\alpha}_{n'm'}\,{}_A<m'|\right) \\ & \otimes |\Phi_\alpha(x_0)><\Phi_\alpha(x_0)|. \end{aligned} \tag{5}$$

However, since the state inside the black hole is not accessible for the outside observer A, we trace over the state of B to obtain a reduced density operator for A and the detector as

$$\begin{aligned} \rho'_{A\Phi} &\equiv \sum_\alpha p_\alpha \rho'_\alpha |\Phi_\alpha(x_0)><\Phi_\alpha(x_0)| \\ &= \sum_\alpha A_\alpha \rho_A A^\dagger_\alpha |\Phi_\alpha(x_0)><\Phi_\alpha(x_0)|, \end{aligned} \tag{6}$$

where $A_\alpha \equiv <\Phi_\alpha(x_0)|U|\Phi_0(x_0)>$. If the outside agent does not 'read' the detector, the detector states in eq. (6) must be traced out and then eq. (2) is reproduced. What we have seen above is an explicit construction of a unitary representation of the POVM where we identify the extended Hilbert space as that including the detector states [13].

Now, the experiment is a local and isothermal process due to the Unruh effect of the accelerated system with the temperature $\bar{T}(r) \equiv T_{BH}/\chi(r)$, the blue shifted temperature from the Hawking temperature T_{BH} of the cavity surrounding the black hole at infinity. The first law of black hole physics is

$$\Delta S_{BH} = \frac{\Delta M}{T_{BH}} = \frac{\Delta W}{T_{BH}}, \tag{7}$$

where ΔW is the work needed for the quantum experiment. In the semi-classical gedanken experiment, this corresponds to the work to push down the box towards the black hole against the buoyancy force by the Hawking radiation [6–8].

Ordinary thermodynamics tells us that the work ΔW needed in the isothermal process is more than or equal to the variation of the free energy:

$$\Delta W \geq \Delta F \tag{8}$$

(with the equality in (8) holding for a quasi-static process), where

$$\begin{aligned} \Delta F &\equiv \sum_\alpha p_\alpha[E_\alpha - \bar{T}(S'_\alpha - \log p_\alpha)]\chi - (E_0 - \bar{T}S_M)\chi \\ &= \left[S_M - \left(\sum_\alpha p_\alpha S'_\alpha - \sum_\alpha p_\alpha \log p_\alpha\right)\right] T_{BH}, \end{aligned} \tag{9}$$

and we have used the conservation of the internal energy $E_0 = \sum_\alpha p_\alpha E_\alpha$, which holds in the isothermal system (E_0 and E_α are the energies of the Hawking state before and after the experiment, respectively). Furthermore, S_M and S'_α are defined by $S_M \equiv S(\rho_A)$ (the initial matter entropy) and $S'_\alpha \equiv S(\rho'_\alpha)$, where $S(\rho) \equiv -\,\mathrm{Tr}\,(\rho \log \rho)$ is the von Neumann entropy for a general state ρ. The last term on the r.h.s. of eq. (9) represents the final entropy of the detector, which reflects our ignorance about the actual outcome of the measurement.

Combining the first law of black hole physics and the second law of thermodynamics given above, we then easily obtain $\Delta S_{BH} = S'_{BH} - S_{BH} \geq S_M - \sum_\alpha p_\alpha(S'_\alpha - \log p_\alpha)$ or, in a more illuminating way,

$$(S'_{BH} + S'_M) - (S_{BH} + S_M) \geq 0 \tag{10}$$

where $S'_M \equiv S(\rho'_{A\Phi}) = \sum_\alpha p_\alpha S'_\alpha - \sum_\alpha p_\alpha \log p_\alpha$ is the matter entropy after the measurement (including the contribution from the detector). In other words, the generalized second law holds.

Let us now extend the previous argument to the case in which the observer disposes of the detector in a gedanken experiment a lá Geroch-Bekenstein. Suppose that the observer conditionally drops the detector into the black hole if the experiment outcome is $\alpha \in D$, while keeping it outside the black hole if $\alpha \notin D$. That is, the detector might alter the state inside the black hole if the measurement outcome $\alpha \in D$. In general the state (5) will change further to

$$\begin{aligned} \sigma' = \sum_\alpha & \left(\sum_n \sqrt{c_n} V_\alpha |n>_B \sum_m U^\alpha_{nm}|m>_A\right) \\ & \cdot \left(\sum_{n'} \sqrt{c_{n'}}\,{}_B<n'|V^\dagger_\alpha \sum_{m'} U^{*\alpha}_{n'm'}\,{}_A<m'|\right) \\ & \otimes |\Phi_\alpha(x_\alpha)><\Phi_\alpha(x_\alpha)|, \end{aligned} \tag{11}$$

where V_α is a nontrivial unitary transformation if the experimental outcome is $\alpha \in D$ and $V_\alpha = 1$ if $\alpha \notin D$. Moreover, x_α is the spacetime point of the detector sufficiently after the measurement: x_α is inside the black hole if $\alpha \in D$ and it is outside otherwise. This corresponds to the "classical communication from Alice to Bob" in the standard quantum communication set-up, except that in the present case it is an inherently one-way communication.

The trace over the B states washes out the V_α dependence altogether and we obtain the reduced density matrix for the compound state of A and the detector as

$$\sigma'_{A\Phi} \equiv \left(\sum_{\alpha\in D} p_\alpha \rho'_\alpha\right)\rho_D + \sum_{\alpha\notin D} p_\alpha \rho'_\alpha |\Phi_\alpha(x_\alpha)><\Phi_\alpha(x_\alpha)|, \qquad (12)$$

where we have introduced the reduced density operator for the detector as $\rho_D \equiv [\sum_{\alpha\in D} p_\alpha |\Phi_\alpha(x_\alpha)><\Phi_\alpha(x_\alpha)|]/p_D$, with $p_D \equiv \sum_{\alpha\in D} p_\alpha$ the total probability that the detector is dropped into the black hole. For $\alpha \in D$ the detector Hilbert space is tensored with the Hilbert space of the outside observer because the detector and the outside observer get causally disconnected and therefore decoupled. It is then straightforward to compute the matter entropy (now $S'_M \equiv S(\sigma'_{A\Phi})$) as

$$S'_M \equiv -p_D \sum_{\alpha\in D} \hat{p}_\alpha \log \hat{p}_\alpha + S\left(p_D \sum_{\alpha\in D} \hat{p}_\alpha \rho'_\alpha\right) + \sum_{\alpha\notin D} p_\alpha S'_\alpha - \sum_{\alpha\notin D} p_\alpha \log p_\alpha, \qquad (13)$$

where $\hat{p}_\alpha \equiv p_\alpha/p_D$ is the normalized probability for $\alpha \in D$.

The change of free energy is still given by eq. (9), and an almost identical argument as before leads to

$$(S'_{BH}+S'_M) - (S_{BH}+S_M) \geq S'_M - \sum_\alpha p_\alpha S'_\alpha + \sum_\alpha p_\alpha \log p_\alpha. \qquad (14)$$

Finally, substituting eq. (13) into eq. (14) we obtain

$$\Delta S_T \equiv (S'_{BH}+S'_M) - (S_{BH}+S_M) \geq p_D\left[S\left(\sum_{\alpha\in D}\hat{p}_\alpha\rho'_\alpha\right) - \sum_{\alpha\in D}\hat{p}_\alpha S'_\alpha\right]. \qquad (15)$$

Now, when the detector is not dropped into the black hole, eq. (15) reduces to eq. (10), i.e the generalized second law holds. On the other hand, in the dropping case we note that the quantity inside the brackets on the right hand side of eq. (15) is the same appearing in the famous Holevo bound [11,12]:

$$H'_D \equiv S\left(\sum_{\alpha\in D}\hat{p}_\alpha\rho'_\alpha\right) - \sum_{\alpha\in D}\hat{p}_\alpha S(\rho'_\alpha) \geq I'_D, \qquad (16)$$

where I'_D is the mutual information of the components $\alpha \in D$ which would be obtained if one performed a further observation. More precisely, with $\{E_j\}$ being the orthogonal projection summing to unity which corresponds to the further observation at infinity and should be distinguished from the previous POVM, one has

$$I'_D(E) = -\sum_{j,\alpha\in D}\hat{p}_\alpha p(j|\alpha)\log\frac{p(j)}{p(j|\alpha)}, \qquad (17)$$

where $p(j|\alpha) \equiv \mathrm{Tr}\,(E_j\rho'_\alpha)$ is the conditional probability to obtain the outcome j when the state ρ'_α is prepared and $p(j) \equiv \sum_{\alpha\in D}\hat{p}(\alpha)p(j|\alpha)$ is the average probability to obtain j. Eq. (17) can be interpreted as the mutual information between the state prepared by an agent near the black hole and that of another agent at infinity, i.e. the uncertainty of the first measurement minus its uncertainty after the second measurement. The equality can be achieved for some projection $\{E_j\}$ if and only if the components of the ρ'_αs are mutually commuting. In this case the ρ'_αs can be simultaneously diagonalized so that we can choose, for example, that $A^\dagger_\alpha A_\alpha = E_j$ as the best that the second agent can do. In this optimal case we obtain $I'_D(E) = -\sum_{\alpha\in D}\hat{p}_\alpha\log\hat{p}_\alpha$, which is nothing but the Shannon information entropy stored by the first measurement. To summarize, eq. (15) tells us that, this potentially acquirable classical information is bounded from above by the change of the generalized entropy, i.e.

$$\Delta S_T \geq p_D I'_D \qquad (18)$$

In the ordinary thermodynamics of a closed system $\Delta W = 0$, so that we have $S'_M - S_M \geq p_D I'_D$: the acquirable information is not more than the change of entropy.

It is also illuminating to consider an ideal case in which the first agent performs a series of successive quasi-static measurements. In the quasi-static isothermal process, the work which is needed under the influence of an inhomogeneous Hamiltonian H in an experiment a lá Stern-Gerlach equals the change of free energy, i.e. $\Delta W = \int \mathrm{Tr}\,[\partial_{\mathbf{r}}H(\mathbf{r})e^{-\beta H(\mathbf{r})}]\cdot d\mathbf{r}/Z = -\beta^{-1}\int\partial_{\mathbf{r}}\log Z\cdot d\mathbf{r} = \Delta F$, where $Z \equiv \mathrm{Tr}\,[e^{-\beta H(\mathbf{r})}]$ and $F \equiv -\beta^{-1}\log Z$. Therefore, the equality is saturated in eq. (18):

$$\Delta S_T = p_D H'_D = p_D\left[S\left(\sum_{\alpha\in D}\hat{p}_\alpha\rho'_\alpha\right) - \sum_{\alpha\in D}\hat{p}_\alpha S'_\alpha\right]. \qquad (19)$$

Noting that $H'_D \equiv \sum_{\alpha\in D}\hat{p}_\alpha S(\rho'_\alpha||\rho')$ and the known fact that in general the relative entropy $S(*||**)$ does not increase by further measurement [13], we see that the amount of increase of the total entropy becomes less and less at each step of measurement and eventually does not change at all. This is reminiscent of Prigogine's theorem on minimum entropy production [15], according to which the entropy production rate should not increase in a steady state linear thermodynamical process approaching equilibrium.

Consider a further ideal situation: a quasi-static orthogonal measurement by the first agent near the black hole followed by the same orthogonal measurement by the second agent at infinity, so that in eq. (18) the equality is doubly saturated, i.e. $\Delta S_T = p_D I'_D = -p_D \sum_{\alpha \in D} \hat{p}_\alpha \log \hat{p}_\alpha$, and a black hole of sufficiently large mass M so that the time scale of evaporation is slow enough compared with that of the quantum measurement. We can then think of the situation where the state σ' is distorted from the thermal state $\rho_0 \equiv |\psi >_{HH}< \psi|$ by the quantum measurement, i.e. $\rho_0 \to \sigma'$, and it relaxes back to the initial thermal state ρ_0, assuming that the whole system is surrounded by a cavity with temperature T_{BH}. When the relaxation $\sigma' \to \rho_0$ eventually occurs, the energy ΔW is emitted to infinity in a form of radiation, and the information I'_D initially stored in the state σ' is encoded in the radiation itself. Thus, the information can be completely retrieved by this relaxation process in the ideal case. Of course, it is possible to drop matter into a black hole without distorting the compound state of A and B. However, in this case the observer cannot get any information so that he has no information to loose. The thermal state remains the thermal state so that the radiation from the black hole does not carry any information.

III. SUMMARY AND DISCUSSION

We have shown that the increase of the generalized entropy by a quantum process outside the horizon of a black hole is more than the Holevo bound of classical mutual information lost into the black hole. What we have used as physics are the energy conservation for an isothermal process in the black hole spacetime and the second law of ordinary thermodynamics. The difference between the ordinary POVMs and those in the black hole spacetime is that the *work* needed for the experiment makes the black hole more massive. One might consider ours as a special and hypothetical gedanken experiment. After a little thought, however, one may realize that this represents a fact of real life. After all black holes exist somewhere in the universe and any physical process can be considered as a POVM outside the black holes. The present argument is universal not only in the sense that POVMs represent the most general physical process including, for example, gas collision before the infall, but also in the sense that the quantum state is entangled for all kinds of particles because gravity is universally coupled to any matter. Of course our discussion does not completely solve the information loss paradox, because our treatment of the black hole is semi-classical. One will need a full theory of quantum gravity to really understand the process of information loss and retrieval after a complete evaporation of the black hole, the final stage of which is expected to be trans-Planckian.

In conclusion, our suggestion is that the information loss paradox is not merely an issue of evolution from pure to mixed states, but rather it should be fully addressed within the context of quantum measurement and information theory.

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